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COMMENT

Comment on 'The Fuchsian differential equation of the square lattice Ising model $\chi^{(3)}$ susceptibility'

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Abstract

The new singularities in $\chi^{(3)}$ found by Zenine *et al* (2004 *J. Phys. A: Math. Gen.* **37** 9651) are pinch-type singularities analogous to those found in Feynman graphs in field theory. A general formula for the possible location of all such singularities for arbitrary $\chi^{(N)}$ is given.

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In two impressive recent papers Zenine *et al* [1] have obtained the linear differential equation whose solution gives the exact 3-particle contribution $\chi^{(3)}$ to the Ising model susceptibility. In addition to confirming known singularities [2], they find an 'unexpected' pair satisfying $1 + 3\omega + 4\omega^2 = 0$ in their temperature variable ω. These are pinch singularities that have a direct analogue in the singularities of the integrals describing Feynman graphs in field theory (cf [3], pp 231–5).

To understand their origin[1] in the context of the $(N-1)$-dimensional integrals over trigonometric phases φ_i defining the general $\chi^{(N)}$ note that the integrand contains branch point singularities all of which are of the form of zeros of $\sqrt{f_i(\omega)}$ where $f_i(\omega) = (\cos\varphi_i - s - 1/s)^2 - 1 = (\cos\varphi_i - 1/2\omega)^2 - 1$. Provided ω is on the physical, i.e. first, sheet of the complex ω plane cut on the real axis for $|\omega| \geqslant 1/4$, the function $f_i(\omega)$ cannot vanish for any real phase φ_i and the branch cuts defined by $\sqrt{f_i(\omega)}$ will not be crossed in integration over φ_i. Now suppose that ω is continued onto some other sheet; in that case, we replace the integrals in the real φ_i by closed curve contour integrals in $z_i = \exp(\mathrm{i}\varphi_i)$. If we can deform the contours such that the zeros of $f_i(\omega)$ are avoided as ω is moved then the $\chi^{(N)}$ remains non-singular. Now any deformation, and hence any change in ω, could be accommodated if it were not for the phase constraint $\Sigma\varphi_i = 0 \bmod 2\pi$. The analogous constraints in Feynman graph evaluation are the relationships imposed on internal, i.e. integration variable, momenta by the given external momenta. In both cases, the result is that the integration contours can get pinched between a pair of singularities and the resulting integral is then singular. The explicit singular situation here is that when n phases are equal and satisfy $\cos\varphi^{(+)} = 1 + 1/2\omega$, the remaining $m = N - n$ equal phases cannot avoid $\cos\varphi^{(-)} = -1 + 1/2\omega$. Furthermore, the phase constraint that in real variables is $\Sigma\varphi_i = 0 \bmod 2\pi$ can be generalized to arbitrary

[1] See [1, 2] for the notation used in the following and also for explicit formulae not reproduced here.

complex phase by the condition $\Pi z_i = 1$. This in turn implies $n\varphi^{(+)} + m\varphi^{(-)} = 0 \bmod 2\pi$ can be written as $\cos(n\varphi^{(+)}) - \cos(m\varphi^{(-)}) = 0$. In summary, pinch singularities in $\chi^{(N)}$ can occur whenever

$$\begin{aligned} &\cos(n\varphi^{(+)}) - \cos(m\varphi^{(-)}) = 0, && m + n = N, \\ &\cos\varphi^{(+)} = 1 + 1/(2\omega), && \cos\varphi^{(-)} = -1 + 1/(2\omega). \end{aligned} \tag{1}$$

Equations (1), which are the analogue of the Landau conditions [3] for Feynman graphs, can be trivially reduced to polynomials in $1/2\omega$ or ω by expanding $\cos(k\varphi)$ in powers of $\cos\varphi$. I emphasize that equations (1) are only necessary conditions for the existence of pinch singularities. They may not occur on any given sheet in the ω plane simply because the contour topology is such that not all $N z_i$ contours traverse the pinch.

The empirical evidence for pinch singularities is as follows. For $N = 2$, (1) cannot be satisfied for any ω in agreement with the known simple structure of $\chi^{(2)}$. For $N = 3$, the two possible m, n combinations in (1) lead to $1 - 5\omega = 0$ and $1 + 3\omega + 4\omega^2 = 0$. Zenine *et al* [1] find the second but not the first confirming that (1) is necessary but not sufficient. For $N = 4$, (1) predicts $1 - 6\omega + 8\omega^2 - 4\omega^3 = 0$ and $1 + 6\omega + 8\omega^2 + 4\omega^3 = 0$. Neither set is found [4, 5]. For $N = 5$, the possible sets of singularities from (1) satisfy $1 - 8\omega + 20\omega^2 - 17\omega^3 = 0$, $1 - 7\omega + 5\omega^2 - 4\omega^3 = 0$, $1 + 5\omega + 13\omega^2 = 0$ or $1 + 8\omega + 20\omega^2 + 15\omega^3 + 4\omega^4 = 0$ but only the last set occurs [4].

One final remark. Equations (1) exhaust the possible singularities of the pinch type but this does not exclude the existence of more singularities of, say, the van Hove type like those found lying on the real axis cuts in ω or equivalently on the unit circle in the s plane where $2\omega = s/(1 + s^2)$.

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References

[1] Zenine N, Boukraa S, Hassani S and Maillard J M 2004 *J. Phys. A: Math. Gen.* **37** 9651
Zenine N, Boukraa S, Hassani S and Maillard J M 2005 *J. Phys. A: Math. Gen.* **38** 1875
[2] Nickel B 1999 *J. Phys. A: Math. Gen* **32** 3889
Nickel B 2000 *J. Phys. A: Math. Gen.* **33** 1693
[3] Bjorken J D and Drell S D 1965 *Relativistic Quantum Fields* (New York: McGraw-Hill)
[4] Maillard J M 2005 private communication
[5] Zenine N, Boukraa S, Hassani S and Maillard J-M 2005 *J. Phys. A: Math. Gen.* **38** 4149–73